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On resolving the multiplicity of the branching rule $GL(2k+1, \mathbb{C}) \downarrow SO(2k+1, \mathbb{C})$

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Abstract. We consider the multiplicity problem of the branching rule $GL(2k+1, \mathbb{C}) \downarrow SO(2k+1, \mathbb{C})$. Finite-dimensional irreducible representations of $GL(2k+1, \mathbb{C})$ are realized as right translations on subspaces of the holomorphic Hilbert (Bargmann) spaces of $q \times (2k+1)$ complex variables. Maps are exhibited which carry an irreducible representation of $SO(2k+1, \mathbb{C})$ into these subspaces. An algebra of commuting operators is constructed. Eigenvalues and eigenvectors of certain of these operators can then be used to resolve the multiplicity in the branching rule.

1. Introduction

One of the outstanding problems in the representation theory of Lie groups is the branching rule problem. Let G be a given Lie group and H a subgroup of G. Then it is well known that when we restrict a finite-dimensional irreducible representation of G to H, the representation can be decomposed as a direct sum of irreducible representations of H, provided that the restriction is completely reducible. The same irreducible representation of H may appear more than once in the decomposition. The branching rule $G \downarrow H$ consists of finding the multiplicity of an irreducible representation of H that occurs in the decomposition. Many mathematicians and physicists have studied branching rules of different classical Lie groups (King 1975, Koike and Terada 1987, Whippman 1965, Zelobenko 1970). These branching rules only give the multiplicity of an irreducible representation of H and do not distinguish the equivalent irreducible representations. Leung (1994) found a canonical way of labelling the equivalent representations that occur in the branching rule $GL(2k, \mathbb{C}) \downarrow Sp(2k, \mathbb{C})$ and broke the multiplicity that appears in the branching rule explicitly. A similar method can be adapted to investigate the branching rule $GL(2k+1, \mathbb{C}) \downarrow SO(2k+1, \mathbb{C})$. In this addendum, we want to label the equivalent representations that appear in this branching rule and break the multiplicity explicitly, assuming the multiplicity is known. The general set-up for our problem will be discussed in section 2. In section 3, using the theory of dual pairs (Howe 1985, Moshinsky and Quesne 1970), we will exhibit a new class of generalized commuting Casimir operators. The eigenvalues and eigenvectors of certain elements in this algebra can then be used as labels to distinguish the equivalent representations that occur in the branching rule.

2. *The general set-up of the problem*

Let G denote the general linear group $GL(2k+1, \mathbb{C})$ and H denote the odd complex orthogonal subgroup $SO(2k+1, \mathbb{C})$. In order to break the multiplicity of the branching rule,

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we need to obtain concrete realizations of finite-dimensional irreducible representations of G and H. Recall that an irreducible representation of G is parametrized by an $(2k+1)$-tuple of non-negative integers $(m) = (m_1, \ldots, m_{2k+1})$, which satisfies the dominant condition $m_1 \geqslant \cdots \geqslant m_{2k+1}$. Suppose $(m) = (m_1, \ldots, m_{2k+1})$ such that $m_{q+1} = \cdots = m_{2k+1} = 0$ for some $1 \leqslant q \leqslant 2k+1$. A concrete realization of a finite-dimensional irreducible representation of G, denoted by $(R_{GL}^{(m)}, V_{GL}^{(m)})$, can be constructed as shown in Klink and Ton-That (1988). An irreducible homomorphically induced representation of H is parametrized by an k-tuple of non-negative integers $(m) = (m_1, \ldots, m_k)$, called the signature, such that $m_1 \geqslant \cdots \geqslant m_k$. A concrete realization of such an irreducible representation of H can be obtained as follows. Let B_k denote the lower triangular subgroup of $GL(k, \mathbb{C})$. We define a holomorphic character

$$\xi^{(m)} : B_k \to \mathbb{C}^*$$

$$\xi^{(m)}(b) = b_{11}^{m_1} \cdots b_{kk}^{m_k} \qquad \forall b \in B_k.$$

Consider the following space:

$$V_H^{(m)} = \Big\{ f : \mathbb{C}^{k\times 2k+1} \to \mathbb{C} | f \text{ polynomial function, } f(bX) = \xi^{(m)}(b) f(x)$$

$$\text{for } b \in B_k, X \in \mathbb{C}^{k\times 2k+1} \text{ and } \sum_{p=1}^{2k+1} \left(\frac{\partial^2 f}{\partial Z_{ip} \partial Z_{jp}} \right) = 0, 1 \leqslant i \leqslant j \leqslant k \Big\}.$$

Let $R_H^{(m)}$ denote the representation of H on $V_H^{(m)}$ by right translation, that is $(R_H^{(m)}(h)f)(Z) = f(Zh)$, $h \in H$. Then according to Ton-That, $R_H^{(m)}$ is irreducible with signature (m).

If D_q denotes the group of all complex diagonal invertible matrices of order q, and if $(M) = (M_1, \ldots, M_q)$ is any q-tuple of non-negative integers, we define a holomorphic character

$$\zeta^{(M)} : D_q \to \mathbb{C}^* \qquad \zeta^{(M)}(d) = d_{11}^{M_1} \ldots d_{qq}^{M_q} \qquad \forall d \in D_q.$$

A polynomial function $p : \mathbb{C}^{q\times 2k+1} \to \mathbb{C}$ is said to transform covariantly with respect to $\zeta^{(M)}$ if $f(dZ) = \zeta^{(M)}(d) f(Z)$, for all (d, Z) belonging to $D_q \times \mathbb{C}^{q\times 2k+1}$. We shall denote this subspace by $P^{(M)}$. Now, suppose $(m) = (m_1, \ldots, m_q)$ is a q-tuple of integers such that $m_1 \geqslant \cdots \geqslant m_q \geqslant 0$. Let $L^{(m)}$ denote the representation of $GL(q, \mathbb{C})$ on $P^{(m)}$ defined by $(L^{(m)}(g)p)(Z) = p(g^{-1}Z)$, $g \in GL(q, \mathbb{C})$ and $R^{(m)}$ denote the representation of G on $P^{(m)}$ by right translation. If we let L_{ij} (respectively, R_{rs}) denote the infinitesimal operators of $L^{(m)}$ (respectively, $R^{(m)}$) corresponding to the standard basis e_{ji} (respectively, e_{rs}) of the Lie algebra $\mathbb{C}^{q\times q}$ (respectively, $\mathbb{C}^{2k+1\times 2k+1}$) of $GL(q, \mathbb{C})$ (respectively, G); then we have

$$L_{ij} = \sum_{\eta=1}^{2k+1} Z_{i\eta} \frac{\partial}{\partial Z_{j\eta}} \qquad R_{rs} = \sum_{\eta=1}^{q} Z_{\eta r} \frac{\partial}{\partial Z_{\eta s}} \qquad 1 \leqslant i, j \leqslant q, 1 \leqslant r, s \leqslant 2k+1$$

and the space $V_{GL}^{(m)}$ consists of polynomial functions in $P^{(m)}$ which are simultaneously annihilated by all lowering operators of the form

$$L_{ij} \qquad \text{with } 1 \leqslant i < j \leqslant q. \tag{2.1}$$

Let us denote by $(R_{GL}^{(m)}|_H, V_{GL}^{(m)})$ the restriction of the representation of $R_{GL}^{(m)}$ to H. The dual pair of reductive groups that is needed in this paper is $(Sp(2q, \mathbb{R}), SO(2k+1, \mathbb{C}))$ (Howe 1985, Moshinsky and Quesne 1970).

Let

$$L_{ij} = \sum_{\eta=1}^{2k+1} Z_{i\eta} \frac{\partial}{\partial Z_{j\eta}} \qquad P_{ij} = \sum_{\eta=1}^{2k+1} Z_{i\eta} Z_{j\eta}$$

and

$$D_{ij} = \sum_{\eta=1}^{2k+1} \frac{\partial^2}{\partial Z_{i\eta} \partial Z_{j\eta}} \qquad 1 \leqslant i, j \leqslant q. \tag{2.2}$$

These operators form a basis for the Lie algebra $sp(2q, \mathbb{R})$ of the group $Sp(2q, \mathbb{R})$. Let $\mathcal{F}$ denote the Fock space of q by $2k+1$ complex variables, as constructed in Leung (1994). Then these operators generate a universal envelopping algebra $\mathcal{U}$ of differential operators which acts on $\mathcal{F}$.

3. The multiplicity breaking of the branching rule $G \downarrow H$

We now give the procedure for breaking the multiplicity that appears in the branching rule $G \downarrow H$. Suppose the H-module $(R_{SO}^{(m')}, V_{SO}^{(m')})$ occurs in $V_{GL}^{(m)}$ μ times (there are many formulae that can be used to compute μ, see Koike and Terada (1987) for example). Then from a consequence of Burnside's theorem and the theory of dual pairs (Howe 1985), there exist μ linearly independent elements in $\mathcal{U}$ which form a basis for the vector space $\mathrm{Hom}_H(V_{SO}^{(m')}, V_{GL}^{(m)})$ of all intertwining operators from $V_{SO}^{(m')}$ to $V_{GL}^{(m)}$. If $h_{\max}^{(m')}$ is the highest weight vector of $V_{SO}^{(m')}$, then one can choose μ elements $p_1, \ldots, p_\mu$ of $\mathcal{U}$ such that $p_i h_{\max}^{(m')}$, $1 \leqslant i \leqslant \mu$, are linearly independent highest weight vectors of the μ copies of the H-module equivalent to $V_{SO}^{(m')}$ which are contained in $V_{GL}^{(m)}$. If $W_{\max}^{(m')(m)}$ denote the vector space spanned by $p_i h_{\max}^{(m')}$, then $W_{\max}^{(m')(m)}$ is equivalent to the subspace of μ copies of H-module $V_{SO}^{(m')}$ in $(R_{GL}^{(m)}|_H, V_{GL}^{(m)})$. In order to break the multiplicity in $(R_{GL}^{(m)}|_H, V_{GL}^{(m)})$, we need to find Hermitian operators in $\mathcal{U}$ which commute with the operators L_{ij} in (2.1) (but without the condition $i < j$) and which decompose $W_{\max}^{(m')(m)}$ into distinct one-dimensional eigenspaces.

To find the operators in $\mathcal{U}$ which commute with L_{ij}, we write L_{ij}, $1 \leqslant i, j \leqslant q$, into a $q \times q$ matrix $[\mathbf{L}]$, that is

$$[\mathbf{L}] = \begin{bmatrix} L_{11} & \ldots & L_{1q} \\ \vdots & \ddots & \vdots \\ L_{q1} & \ldots & L_{qq} \end{bmatrix}. \tag{3.1}$$

Similarly, we write P_{ij} (respectively, D_{ij}), $1 \leqslant i, j \leqslant q$, into a $q \times q$ matrix $[\mathbf{P}]$ (respectively, $[\mathbf{D}]$). Also, let $[\mathbf{E}]$ denote the matrix $[-\mathbf{L}]^{\mathrm{T}}$, where T denotes the transpose. Notice that $[\mathbf{P}]$ and $[\mathbf{D}]$ are symmetric matrices. Then we have the following theorem.

Theorem 3.1. In the universal enveloping algebra $\mathcal{U}$, consider the trace of arbitrary products of the following matrices:

(i) $[\mathbf{L}]$
(ii) $[\mathbf{P}][\mathbf{D}]$ and (3.2)
(iii) $[\mathbf{P}][\mathbf{E}][\mathbf{D}]$.

Then these operators generate a subalgebra $\mathcal{V}$ of differential operators in $\mathcal{U}$ that commute with the operators L_{ij}, $1 \leqslant i, j \leqslant q$.

Example. We can form the following commuting operator:

$$\mathrm{tr}([\mathbf{P}][\mathbf{E}][\mathbf{D}][\mathbf{L}]).$$

We call the algebra $\mathcal{V}$ an algebra of generalized Casimir operators. Now, let $\mathbf{R}$ denote the matrix (R_{rs}) and we have the following theorem.

Theorem 3.3. The differential operators of the form $\mathrm{tr}(\mathbf{A}_1 \ldots \mathbf{A}_i \ldots \mathbf{A}_r)$, where $\mathbf{A}_i = \mathbf{R}$ or $\mathbf{R}^{\mathrm{T}}$, $1 \leqslant i \leqslant r$, r is an integer $\geqslant 0$, generate the same algebra $\mathcal{V}$ of commuting differential operators as the differential operators defined by (3.2) in theorem 3.1. The adjoint of such an operator $\mathrm{tr}(\mathbf{A}_1 \ldots \mathbf{A}_r)$ is given by $\mathrm{tr}(\mathbf{A}_r \ldots \mathbf{A}_1)$.

The proofs of the above theorems are similar to that for theorem 3.3, proposition 3.5, 3.7 in Leung (1994). From theorem 3.3 above, the adjoint of an operator in $\mathcal{V}$ is still in $\mathcal{V}$. If an operator is not Hermitian, then the sum of the operator and its adjoint would be Hermitian. Hence, we can always find a Hermitian operator in $\mathcal{V}$. All we need to do now is to pick a Hermitian Casimir operator $\mathcal{C}$ in $\mathcal{V}$ and use it to decompose the $W_{\max}^{(m')(m)}$ space into distinct one-dimensional subspaces. Recall that $(R_{SO}^{(m')}, V_{SO}^{(m')})$ occurs in $V_{GL}^{(m)}$ μ times. After we apply the Casimir operator $\mathcal{C}$ on $W_{\max}^{(m')(m)}$, we obtain μ distinct eigenvalues. The eigenvectors from all one-dimensional subspaces are orthogonal to each other with respect to the inner product of the Fock space $\mathcal{F}$ because the eigenvalues are all distinct. Since the Casimir operator $\mathcal{C}$ commutes with operators L_{ij}, it leaves the space $V_{GL}^{(m)}$ invariant and $\mathcal{C}(W_{\max}^{(m')(m)})$ is a subspace in $V_{GL}^{(m)}$. Therefore, the eigenvectors can be used as labels to distinguish the equivalent representations that appear more than once in the branching rule. In the next section, we shall illustrate the procedure by an example.

4. An illustration

In this section, we are going to demonstrate briefly the procedure outlined above by considering the multiplicity breaking of the irreducible representation $(1, 1)$ of $SO(5, \mathbb{C})$ in the irreducible representation $(4, 2, 0, 0, 0)$ of $GL(5, \mathbb{C})$ when we restrict this representation to the subgroup $SO(5, \mathbb{C})$. According to a result in Koike and Terada (1987), the irreducible representation $(1, 1)$ of $SO(5, \mathbb{C})$ occurs in this restriction twice.

According to our programme, the dual pair for this example is $(sp(4, \mathbb{R}), SO(5, \mathbb{C}))$. Let $h_{\max}$ be the highest weight vector of $V_H^{(1,1)}$, then it is given by

$$h_{\max} = z_{11}z_{22} - z_{12}z_{21}.$$

Since the H-module $(R_H^{(1,1)}, V_H^{(1,1)})$ occurs in $V_{GL}^{(4,2,0,0,0)}$ twice, we need to pick two linearly independent intertwining operators, p_1 and p_2, in $\mathcal{U}(sp(4, \mathbb{R}))$ that send the H-module $V_H^{(1,1)}$ into the $GL(5, \mathbb{C})$-module $V_{GL}^{(4,2,0,0,0)}$. We want to mention how to choose the two linearly independent operators. Our goal is to find elements in $\mathcal{U}(sp(4, \mathbb{R}))$ that send $V_H^{(1,1)}$ into the $GL(5, \mathbb{C})$-module $V_{GL}^{(4,2,0,0,0)}$. In $\mathcal{U}(sp(4, \mathbb{R}))$, the raising operators are $P_{\alpha\beta}$ and $L_{\alpha\beta}$ for $\alpha > \beta$ and the lowering operators are $P_{\alpha\beta}$ and $D_{\alpha\beta}$ for $\alpha < \beta$. Therefore, we want to combine certain raising and lowering operators in $\mathcal{U}(sp(4, \mathbb{R}))$ so that we can raise the 2-tuple of integers $(1, 1)$ to $(4, 2, 0, 0, 0)$. Now, $p_1 h_{\max}$ and $p_2 h_{\max}$ span the vector space $W_{\max}^{(1,1)(4,2,0,0,0)}$.

In order to break the multiplicity, we need to pick a Hermitian Casimir operator $\mathcal{C}$ in $\mathcal{V}$ and use it to decompose the $W_{\max}^{(1,1)(4,2,0,0,0)}$ space into distinct one-dimensional subspaces. In general, we choose the Casimir operator in an *ad hoc* manner. However, in practice, just a low-degree operator generally suffices, for example, $\mathrm{tr}(\mathbf{R}\mathbf{R}^{\mathrm{T}}\mathbf{R})$ or $\mathrm{tr}(\mathbf{R}\mathbf{R}^{\mathrm{T}})+\mathrm{tr}(\mathbf{R}^{\mathrm{T}}\mathbf{R})$, where

$$R_{rs}=\sum_{\eta=1}^{2} Z_{\eta r}\frac{\partial}{\partial Z_{\eta s}} \qquad 1\leqslant r,\ s\leqslant 5$$

(according to theorem 3.3, they are both Hermitian operators). The Casimir operator $\mathcal{C}$ acting on $W_{\max}^{(1,1)(4,2,0,0,0)}$ will have two distinct eigenvalues λ_1 and λ_2. If the corresponding eigenvector for λ_1 is h_1 and the corresponding eigenvector for λ_2 is h_2, then the eigenvectors h_1 and h_2 are orthogonal because $\mathcal{C}$ is Hermitian and they can be used as labels to distinguish the equivalent representations $V_H^{(1,1)}$ in the restriction.

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